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Sampling theory associated with q-difference equations of the Sturm–Liouville type

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Abstract
We will show that if the kernel $K(x,t)$ in the representation $f(t) = \int_0^a K(x,t)u(t)\,\mathrm{d}_q t$, with $u \in L_q^2(0,a)$, is a solution of a second-order q-Sturm–Liouville boundary problem, then f admits a representation as a sampling formula of the form $f(t) = \sum_{n=0}^{\infty} f(\lambda_n)W(t)/[W'(\lambda_n)(t-\lambda_n)]$, where λ_n is the nth eigenvalue of the associated q-Sturm–Liouville boundary problem and $W(t)$ is the q-Wronskian of two solutions selected in a specified way.

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1. Introduction

The celebrated Whittaker–Shannon–Kotel'nikov theorem states that every integral transform of the form

$$f(x) = \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}xt} u(t)\,\mathrm{d}t,$$

where $u \in L^2(-\pi,\pi)$, can be written as the sampling formula

$$f(x) = \sum_{n=-\infty}^{\infty} f(n)\frac{\sin\pi(x-n)}{\pi(x-n)}. \tag{1}$$

Writing

$$L(x) = \sin\pi z = \pi z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)$$

then the Whittaker–Shannon–Kotel'nikov theorem is a Lagrange-type interpolation formula of the form

$$f(x) = \sum_{n=-\infty}^{\infty} f(n)\frac{L(x)}{L'(n)(x-n)}. \tag{2}$$

The quest of writing sampling theorems as Lagrange-type interpolation formulae has attracted a considerable amount of research over the years. When the kernel e^{ixt} is replaced by some function in the context of Kramer's lemma, then often there are special function formulae available to perform the task. This is also the case of the functions orthogonal with respect to their own zeros [3]. Another interesting situation occurs when the kernel of the integral transform arises from a second-order differential equation. A major achievement in this direction is due to Zayed, Hinsen and Butzer [20]. They considered the second-order Sturm–Liouville eigenvalue as

$$-y''(x) + v(x)y(x) = \lambda y(x), \tag{3}$$

where $v(x)$ is defined on the finite interval $[a, b]$ together with the initial conditions

$$\cos\alpha y(a) + \sin\alpha y'(a) = 0 \qquad \cos\beta y(b) + \sin\beta y'(b) = 0$$

and selected a particular solution of the problem satisfying $\varphi(a, \lambda) = \sin\alpha$ and $\varphi'(a, \lambda) = -\cos\alpha$ and another one satisfying $\psi(b, \lambda) = \sin\beta$ and $\psi'(b, \lambda) = -\cos\beta$. Within this setting, their main theorem reads as follows:

Theorem A. *Every function f that can be written as an integral transform of the form*

$$f(\lambda) = \int_a^b u(x)\varphi(x, \lambda)\,\mathrm{d}x,$$

where $u \in L^2(a, b)$ admits a sampling representation of the form

$$f(\lambda) = \sum_{n=0}^{\infty} f(\lambda_n)\frac{W(\lambda)}{W'(\lambda_n)(\lambda - \lambda_n)},$$

where $W(\lambda)$ is the Wronskian of the functions $\varphi(x, \lambda)$ and $\psi(x, \lambda)$.

The construction of expansions in q-Fourier series ([8, 9]) was followed by the derivation of the q-sampling theorems ([2, 5, 14]). The most relevant feature present in all of these q-sampling theorems is the sparsity of their sampling nodes, located at the zeros of the q-analogues of the $\sin x$. Recent research about these zeros ([1, 18]) indicates that, for big n, they behave very much like sequences of the form q^{-n}. Therefore, the resulting sampling expansions provide a process of reconstructing signals from samples that become sparse as they move away from the origin. As an instance, every function f within the setting of our main result in [2] contains all its information on the sequence $\{f(q^{-n+\epsilon_n})\}_{n\in\mathbf{N}}, 0 < \epsilon_n < 1, 0 < q < 1$. That is, whereas the sampling theorems of the Whittaker–Shannon–Kotel'nikov type identify a function with its values over an arithmetic sequence (or close to such a sequence, as in the case of irregular sampling), the q-sampling theorems identify a function with its values over a geometric sequence. The other novel aspect of the q-sampling theorems is the presence of the parameter q itself. For every realization of the parameter we have a different sampling theory, with different sampling nodes. The idea of using q-deformations of classical results has showed before to be a successful one in physical mathematics as can be testified by the considerable amount of research about the q-harmonic oscillator that followed [7] or the surge of the theory of quantum groups and their connections to q-special functions [10].

For a comprehensive introduction to the subject of sampling from a classic point of view we refer to [13]. For an exposition of modern sampling methods and how they were inspired by physical problems in communication, astronomy and medicine, we suggest the reading of [4], where algorithms for practical reconstruction of signals from samples also are included.

The purpose of the present paper is to construct a q-analogue of theorem A, building on recent results of Annaby and Mansour [6], who provided a detailed study of the basis properties of solutions of q-Sturm–Liouville systems, inspired by the formal work of Exton [11].

2. Preliminaries on q-calculus

Following the standard notation in [12], consider a number $0 < q < 1$ and define the q-shifted factorial for n finite and different from zero as

$$(a; q)_n = (1 - q)(1 - aq) \cdots (1 - aq^{n-1})$$

and the zero and infinite cases as

$$(a; q)_0 = 1, \qquad (a; q)_\infty = \lim_{n\to\infty} (a; q)_n.$$

The q-difference operator D_q is defined as

$$D_q f(x) = \frac{f(x) - f(qx)}{x(1 - q)}.$$

When required q will be replaced by q^{-1}. The following facts can be verified directly from the definition and will be used often:

$$D_{q^{-1}} f(x) = (D_q f)(xq^{-1}), \qquad D_q^2 f(q^{-1}x) = q D_q[D_q f(q^{-1}x)] = D_{q^{-1}} D_q f(x).$$

Associated with this operator there is a nonsymmetric formula for the q-differentiation of a product

$$D_q[f(x)g(x)] = f(qx)D_q g(x) + g(x)D_q f(x). \tag{4}$$

The q-integral usually associated with the name of Jackson is defined, in the interval $(0, a)$, as

$$\int_0^a f(x)\, \mathrm{d}_q x = (1 - q) \sum_{n=0}^{\infty} f(aq^n)aq^n.$$

Let $L_q^2(0, a)$ be the space of all complex-valued functions defined on $(0, a)$, such that

$$\|f\| = \left[\int_0^a |f(x)|^2\, \mathrm{d}_q x\right]^{\frac{1}{2}} < \infty.$$

The space $L_q^2(0, a)$ is a separable Hilbert space (see [5] for details) with the inner product

$$\langle f, g\rangle = \int_0^a f(x)\overline{g(x)}\, \mathrm{d}_q x.$$

Through the remainder of the text we will deal only with functions q-regular at zero, that is, functions such that

$$\lim_{n\to\infty} f(q^n x) = f(0).$$

The class of the functions which are q-regular at zero includes the continuous functions. An example of a function that is not q-regular at zero is given by [17]

$$f(x) = \sin(\alpha \log x) \qquad \text{with} \qquad \alpha \log q = 2\pi.$$

If f and g are both q-regular at zero, there is a rule of q-integration by parts given by

$$\int_0^a g(x)D_q f(x)\, \mathrm{d}_q x = (fg)(a) - (fg)(0) - \int_0^a D_q g(x) f(qx)\, \mathrm{d}_q x. \tag{5}$$

The q appearing in the argument of f in the right-hand side integrand is another manifestation of the asymmetry that is everywhere present in q-calculus. As an important special case, we have

$$\int_0^a D_q f(x)\,\mathrm{d}_q x = (f)(a) - (f)(0). \tag{6}$$

For these and other formulae, we refer to [15].

3. The q-Sturm–Liouville problem

We will consider a q-Sturm–Liouville equation of the form

$$-\frac{1}{q} D_{q^{-1}} D_q y(x) + v(x) y(x) = \lambda y(x) \qquad 0 \leqslant x \leqslant a < \infty, \quad \lambda \in \mathbb{C}. \tag{7}$$

with $v(x)$ defined on the interval $[0, a]$, together with the initial conditions

$$a_{11} y(0) + a_{12} D_{q^{-1}} y(0) = 0 \tag{8}$$

$$a_{21} y(a) + a_{22} D_{q^{-1}} y(a) = 0. \tag{9}$$

It was shown in [6] that such a q-Sturm–Liouville problem is formally self adjoint, that is, denoted by ℓy, the left-hand member of (7), they proved that $\langle \ell y, h\rangle = \langle y, \ell h\rangle$. This self-adjointness property allowed the authors to use the spectral theorem for compact self-adjoint operators, after turning the q-difference problem into a q-integral one, by means of the construction of a q-type Green's function. A key step for the construction of a q-type Green's function in [6] was the definition of a fundamental set of solutions of (7) by means of the q-analogues of the functions sin and cos defined as

$$\sin(x; q) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{n^2}}{(q; q)_{2n}} (x\,(1-q))^{2n} \tag{10}$$

and

$$\cos(x; q) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)}}{(q; q)_{2n+1}} (x(1-q))^{2n+1}. \tag{11}$$

These functions differ slightly from the ones considered in [9]. However, the crucial information about their roots can be obtained by relating them to the third Jackson q-Bessel function

$$J_\nu(x; q) = x^\nu \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)/2}}{(q^{\nu+1}; q)_n (q; q)_n} x^{2n}.$$

Clearly, we have the relations

$$\sin(x; q) = \frac{(q^2; q^2)}{(q^3; q^2)} (x(1-q))^{\frac{1}{2}} J_{\frac{1}{2}}(x(1-q); q^2),$$

$$\cos(x; q) = \frac{(q^2; q^2)}{(q; q^2)} (x(1-q))^{\frac{1}{2}} J_{-\frac{1}{2}}(x(1-q) q^{\frac{1}{2}}; q^2).$$

Combining these relations with the bounds obtained in [1] for the roots of the third q-Bessel function, we obtain the following theorem on the location of the roots of the functions $\sin(x; q)$ and $\cos(x; q)$.

Theorem B. *If* $q < (1-q^2)^2$ *then the nonzero roots of the function* $\sin(x;q)$ *are of the form*

$$x_n = \frac{q^{-n+\epsilon_n}}{1-q} \tag{12}$$

and if $q^3 < (1-q^2)^2$ *then the roots of the function* $\cos(x;q)$ *are of the form*

$$y_n = \frac{q^{-n+1+\epsilon_n}}{1-q}, \tag{13}$$

where $0 < \epsilon_n < 1$. *The restrictions on* q *can be removed if* n *is big enough.*

Remark. It can be seen from the rapid growth of their zeros, or directly from the power series expansions (10) and (11), that both $\sin(x;q)$ and $\cos(x;q)$ are functions of order zero, when considered as entire functions. As a result they are unbounded on the real line (if they were bounded on the real line then we could use a Phragmén-Lindelöf argument to extend the bound to the whole complex plane and this would force the functions to be constant). This unboundedness property constitutes a serious obstacle in obtaining asymptotics for the eigenvalues and eigenfunctions of the q-Sturm–Liouville problem.

We will use the next two results from [6] in our discussion.

Theorem C. *Concerning the above definitions, the following propositions hold:*

(i) Given $c_1, c_2 \in C$, *equation (7) has a unique solution* ϕ, q*-regular at zero and satisfying*

$$\phi(0,\lambda) = c_1, \qquad D_{q^{-1}}\phi(0,\lambda) = c_2, \qquad \lambda \in \mathbb{C}.$$

Moreover, $\phi(x,\lambda)$ *is entire in* λ *for all* $x \in [0,a]$, *where the* Dq^{-1} *derivative of a function* $f(x)$ *at zero is given by*

$$D_{q^{-1}} f(x) = \lim_{n\to-\infty} \frac{f(xq^{-n}) - f(0)}{xq^{-n}} = D_q f(0).$$

Theorem D. *The eigenvalues of the problem (7)–(9) form an infinite sequence of real numbers which can be ordered in an ascending way. Moreover, the set of all normalized eigenfunctions of (7)–(9) forms an orthonormal basis for* $L_q^2(0,a)$.

Essential in our discussion will be the q-Wronskian of two functions f and g defined as

$$W_q(f,g)(x) = f(x)D_q g(x) - g(x)D_q f(x). \tag{14}$$

It was proved by Meijer and Swarttouw that $\{f, g\}$ forms a complete set of solutions of (7) if and only if their q-Wronskian does not vanish at any point of $[0,a]$. The q-Wronskian of a q-Sturm–Liouville problem will play a fundamental role in the next section.

4. The q-sampling theory

In this section we will establish our main result. The key ingredient will be Kramer's lemma, discovered in [16]. It is usually stated with the Lebesgue measure dx but a q-version can be derived without modifying the structure of the proof. Actually, the result can be stated in a very general way, using the inner product in a general Hilbert space. Since it is clear that the q-integral defines a inner product in a Hilbert space, we simply state Kramer's lemma in the required form, and appeal to [13] for more information on the subject.

Theorem E. *(Kramer's lemma). Let* $I \subset R$ *be a bounded interval,* $K(x,t)$ *be a kernel belonging to* $L^2(I)$ *for each fixed* t *in a suitable subset* D *of* R. *Suppose also that, for some* *sequence of points belonging to D, $\{K(x,\lambda_n)\}$ is an orthogonal basis for $L^2(I)$. Under these conditions, every function f written in the form*

$$f(t) = \int_I g(x)K(x,t)\,\mathrm{d}_q x$$

admits the sampling expansion

$$f(t) = \sum_{n=0}^{\infty} f(\lambda_n)\frac{\int_I \overline{K(x,\lambda_n)}K(x,t)\,\mathrm{d}_q x}{\int_I |K(x,t)|^2\,\mathrm{d}_q x};$$

the sampling series converges absolutely and uniformly on every set $C \subset D$ for which $\|K(,t)\|$ is bounded.

Before the proof of our main result, we need to establish some basic facts about the q-Wronskian of q-Sturm–Liouville problems.

Lemma 1. *Let f and g be q-regular at zero. The Wronskian $W_q(f,g)(x)$ of the q-Sturm–Liouville problem (7) does not depend on x.*

Proof. Applying formula (5) in the proper order, we obtain

$$\begin{aligned} D_q W_q(f,g)(x) &= D_q\left[f(x)(D_q g)(x) - g(x)(D_q f)(x)\right] \\ &= f(qx)D_q^2 g(x) - g(qx)D_q^2 f(x). \end{aligned}$$

Then

$$\begin{aligned} D_q W_q(f,g)(q^{-1}x) &= f(x)\big(D_q^2 g\big)(q^{-1}x) - g(x)\big(D_q^2 f\big)(q^{-1}x) \\ &= f(x)[v(x)g(x) - \lambda g(x)] - g(x)[v(x)f(x) - \lambda f(x)] = 0. \end{aligned}$$

As a result,

$$0 = D_q W_q(f,g)(q^{-1}x) = \frac{W_q(f,g)(q^{-1}x) - W_q(f,g)(x)}{q^{-1}x(1-q)}$$

or, for every $x \neq 0$,

$$W_q(f,g)(x) = W_q(f,g)(q^{-1}x).$$

Iterating gives

$$W_q(f,g)(q^n x) = W_q(f,g)(x)$$

for every $n = 1, 2, \ldots$. Taking the limit when $n \to \infty$, we conclude that $W_q(f,g)(x) = W_q(f,g)(0)$, since $W_q(f,g)(x)$ is q-regular at zero. $\square$

From now on, we will invoke theorem C and choose, from the solutions of (7), a particular solution $\varphi(x,\lambda)$, such that it is an entire function of λ, real valued when λ is real and satisfying

$$\varphi(0,\lambda) = -a_{12}, \qquad D_{q^{-1}}\varphi(0,\lambda) = a_{11} \tag{15}$$

and another one satisfying

$$\psi(a,\lambda) = -a_{22}, \qquad D_{q^{-1}}\psi(a,\lambda) = a_{21}, \tag{16}$$

since the q-Wronskian is independent of x, we can evaluate it at $x = a$, and use the above conditions on ψ in order to write

$$W_q(\varphi,\psi)(\lambda) = W(\lambda) = a_{21}\varphi(a,\lambda) + a_{22}D_{q^{-1}}\varphi(a,\lambda). \tag{17}$$

It follows from the initial conditions (9) that $W(\lambda) = 0$ if and only if λ is an eigenvalue of the q-Sturm–Liouville problem. The set-up is now complete to state our main result. The proof will go along the same lines of Zayed's proof in [21].

Theorem 1. *Let $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ be the solutions of (7) selected as above. Then every function f of the form*

$$f(\lambda) = \int_0^a u(x)\varphi(x,\lambda)\,\mathrm{d}_q x, \qquad u \in L^2(0,a) \tag{18}$$

can be written as the Lagrange-type sampling expansion

$$f(\lambda) = \sum_{n=0}^{\infty} f(\lambda_n)\frac{W(\lambda)}{W'(\lambda_n)(\lambda-\lambda_n)}, \tag{19}$$

where $W(\lambda)$ is the q-Wronskian of the functions $\varphi(x,\lambda)$ and $\psi(x,\lambda)$.

Proof. Multiply equation (7) by $\varphi(x,\lambda_n)$. Then consider again equation (7), but replace λ by λ_n and multiply this last equation by $\varphi(x,\lambda)$. Subtracting the two results yields

$$(\lambda-\lambda_n)\,\varphi(x,\lambda)\varphi(x,\lambda_n) = D_q^2\varphi(q^{-1}x,\lambda_n)\varphi(x,\lambda) - D_q^2\varphi(q^{-1}x,\lambda)\varphi(x,\lambda_n)$$

an application of the rule for the q-differentiation of a product (4) gives, choosing the right order in both q-differentiations,

$$= D_q[D_q\varphi(q^{-1}x,\lambda_n)\varphi(x,\lambda) - D_q\varphi(q^{-1}x,\lambda)\varphi(x,\lambda_n)].$$

Performing a q-integration by means of (6) gives

$$\begin{aligned}(\lambda-\lambda_n)&\int_0^a \varphi(x,\lambda)\varphi(x,\lambda_n)\,\mathrm{d}_q x\\ &= D_q\varphi(q^{-1}a,\lambda_n)\varphi(a,\lambda) - D_q\varphi(q^{-1}a,\lambda)\varphi(a,\lambda_n)\\ &\quad - D_q\varphi(0,\lambda_n)\varphi(0,\lambda) - D_q\varphi(0,\lambda)\varphi(0,\lambda_n)\\ &= \varphi(a,\lambda)D_{q^{-1}}(a,\lambda_n) - \varphi(a,\lambda_n)D_{q^{-1}}\varphi(a,\lambda)\end{aligned}$$

the justification for the last identity lies on the fact that by (15) and the initial conditions (8), we have

$$\begin{aligned}D_q\varphi(0,\lambda_n)&\varphi(0,\lambda) - D_q\varphi(0,\lambda)\varphi(0,\lambda_n)\\ &= -D_{q^{-1}}\varphi(0,\lambda_n)a_{12} - a_{11}\varphi(0,\lambda_n) = 0.\end{aligned}$$

Now assume that $a_{21} \neq 0$. Multiply (17) by $\varphi(a,\lambda_n)$ to obtain

$$W(\lambda)\varphi(a,\lambda_n) = a_{21}\varphi(a,\lambda)\varphi(a,\lambda_n) + a_{22}D_{q^{-1}}\varphi(a,\lambda)\varphi(a,\lambda_n).$$

Using the initial condition (8), this identity becomes

$$\begin{aligned}W(\lambda)\varphi(a,\lambda_n) &= -a_{22}D_{q^{-1}}\varphi(a,\lambda)\varphi(a,\lambda) + a_{22}D_{q^{-1}}\varphi(a,\lambda)\varphi(a,\lambda_n)\\ &= a_{22}[\varphi(a,\lambda)D_{q^{-1}}(a,\lambda) - \varphi(a,\lambda_n)D_{q^{-1}}\varphi(a,\lambda)]\end{aligned}$$

as a result,

$$(\lambda-\lambda_n)\int_0^a \varphi(x,\lambda)\varphi(x,\lambda_n)\,\mathrm{d}_q x = \frac{W(\lambda)\varphi(a,\lambda_n)}{(\lambda-\lambda_n)a_{22}}$$

and taking the limit as $\lambda \to \lambda_n$ gives

$$\int_0^a |\varphi(x,\lambda_n)|^2\,\mathrm{d}_q x = W'(\lambda_n)\frac{\varphi(a,\lambda_n)}{a_{21}}.$$

Now, by theorem D, $\{\varphi(x,\lambda_n)\}$ forms an orthogonal basis of $L^2(0,a)$. We can therefore apply Kramer's lemma and write an integral transform of the form (18) as

$$f(\lambda) = \sum_{n=0}^{\infty} f(\lambda_n)\frac{W(\lambda)}{W'(\lambda_n)(\lambda-\lambda_n)}. \tag{20}$$

Consider now the case $a_{21} = 0$. Multiply the identity (17) by $D_{q^{-1}}\varphi(a, \lambda_n)$ to obtain

$$W(\lambda) D_{q^{-1}}\varphi(a, \lambda_n) = a_{22}\varphi(a, \lambda) D_{q^{-1}}\varphi(a, \lambda_n)$$

on the other side, by (9) we have $D_{q^{-1}}\varphi(a, \lambda) = 0$, so that

$$(\lambda - \lambda_n) \int_0^a \varphi(x, \lambda)\varphi(x, \lambda_n)\, \mathrm{d}_q x = \varphi(a, \lambda) D_{q^{-1}}\varphi(a, \lambda_n),$$

we conclude that

$$\int_0^a \varphi(x, \lambda)\varphi(x, \lambda_n)\, \mathrm{d}_q x = \frac{W(\lambda) D_{q^{-1}}\varphi(a, \lambda_n)}{a_{22}(\lambda - \lambda_n)}$$

taking the limit as $\lambda \to \lambda_n$,

$$\int_0^a |\varphi(x, \lambda_n)|^2\, \mathrm{d}_q x = W'(\lambda_n) \frac{D_{q^{-1}}\varphi(a, \lambda_n)}{(\lambda - \lambda_n) a_{22}}$$

and as before, the use of Kramer's lemma gives (20). □

Example. Consider the problem

$$-\frac{1}{q} D_{q^{-1}} D_q y(x) = \lambda y(x)$$

and the conditions

$$y(0) = 0 \qquad y(1) = 1.$$

A fundamental set of solutions is $\{\cos(\sqrt{\lambda}x; q), \sin(\sqrt{\lambda}x; q)\}$. From this we select $\varphi(x, \lambda) = \cos(\sqrt{\lambda}x; q)$ to satisfy the initial conditions. The eigenvalues $\{\lambda_n\}$ of the problem are the zeros of $\sin(\sqrt{\lambda}; q)$, and from (12), if $q < (1 - q^2)^2$ or if n is big enough, then $\lambda_n = (1 - q)^{-2} q^{-2n+\epsilon_n}$. Since the q-Wronskian is a function of order zero and its zeros are the λ_n´s, it follows from theorem 1 that every function of the form

$$f(t) = \int_0^1 u(x) \cos(\sqrt{\lambda}x; q)\, \mathrm{d}_q x$$

has the representation

$$f(\lambda) = \sum_{n=0}^{\infty} f((1 - q)^{-2} q^{-2n+\epsilon_n}) \frac{\sin(\sqrt{\lambda}; q)}{[\sin'(\sqrt{x}; q)]_{x=\lambda_n} (\lambda - (1 - q)^{-2} q^{-2n+\epsilon_n})}.$$

Examples 2 and 3 from [5] can be treated in a similar way.

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References

[1] Abreu L D, Bustoz J and Cardoso J L 2003 The roots of the third Jackson q-Bessel function *Int. J. Math. Math. Sci.* **67** 4241–8

[2] Abreu L D 2005 A q-Sampling theorem related to the q-Hankel transform *Proc. Am. Math. Soc.* **133** 1197–203

[3] Abreu L D Functions q-orthogonal with respect to their own zeros *Proc. Am. Math. Soc.* at press (*Preprint* available at http://www.mat.uc.pt/preprints/ps/p0432.pdf)

[4] Aldroubi A and Gröchenig K 2001 Nonuniform sampling and reconstruction in shift-invariant spaces *SIAM Rev.* **43** 585–620

[5] Annaby M H 2003 q-type sampling theorems *Result. Math.* **44** 214–25

[6] Annaby M H and Mansour Z S 2005 Basic Sturm–Liouville problems *J. Phys. A: Math. Gen.* **38** 3775–97

[7] Arik M and Coon D 1976 Hilbert spaces of analytic functions and generalized coherent states *J. Math. Phys.* **17** 524–7

[8] Bustoz J and Suslov S 1998 Basic analogues of Fourier series on a q-quadratic grid *Meth. Appl. Anal.* **5** 1–38

[9] Bustoz J and Cardoso J L 2001 Basic analogues of Fourier series on a q-linear grid *J. Approx. Theory* **112** 134–57

[10] Chari V and Pressley A 1994 *A Guide to Quantum Groups* (Cambridge: Cambridge University Press)

[11] Exton H 1983 *q-Hypergeometric Functions and Applications* (New York: Ellis Horwood)

[12] Gasper G and Rahman M 1990 *Basic Hypergeometric Series* (Cambridge: Cambridge University Press)

[13] Higgins J R 1996 *Sampling Theory in Fourier and Signal Analysis: Foundations* (Oxford: Clarendon) chapter xiii pp 222

[14] Ismail M E H and Zayed A I 2003 A q-analogue of the Whittaker–Shannon–Kotel'nikov sampling theorem *Proc. Am. Math. Soc.* **131** 3711–9

[15] Kac V and Cheung P 2002 *Quantum Calculus* (Berlin: Springer)

[16] Kramer H P 1959/60 A generalized sampling theorem *J. Math. Phys.* **38** 68–72

[17] Meijer H G and Swarttouw R F 1994 A q-analogue of the Wronskian and a second solution of the Hahn–Exton q-Bessel difference equation *Proc. Am. Math. Soc.* **120** 855–64

[18] Suslov S 2003 Asymptotics of zeros of basic sine and cosine functions *J. Approx. Theory* **121** 292–335

[19] Titchmarsh E C 1962 *Eigenfunctions Expansions Associated with Second-order Differential Equations* (London: Oxford University Press)

[20] Zayed A I, Hinsen G and Butzer P 1990 On Lagrange interpolation and Kramer-type sampling theorems associated with Sturm–Liouville problems *SIAM J. Appl. Math.* **50** 893–909

[21] Zayed A I 1991 On Kramer's sampling theorem associated with general Sturm–Liouville problems and Lagrange interpolation *SIAM J. Appl. Math.* **51** 575–604